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OPTIMAL CONTROL OF MOTIONS OF A BIFILAR PENDULUM†

L. D. AKULENKO

Moscow

e-mail: kumak@ipmnet.ru

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The controlled oscillatory and rotational motions of a rigid body on a plane-parallel bifilar suspension are investigated. The controlled object, the relative position of which can be regulated, is connected to the body. The vectors of the acceleration or the rate of displacement of the object with respect to the body. The vectors of the acceleration or the rate of displacement of the object with respect to the body are used as the control functions. The values of the control functions are assumed to be small compared with the gravitational forces, which enables a small parameter to be introduced into the dimensionless variables. Specific regions of constraints (a rectangle, ellipse, or an inclined segment) are considered. Using asymptotic methods, a solution of the first approximation is constructed for the problem of the optimal control of the oscillation and rotation energies of the system. The case of small oscillations and fast rotations are investigated separately. The qualitative features of the controlled motions of a bifilar pendulum are established and commented on.

The construction, investigation and optimization of controlled motions for pendulum-type rotational-oscillatory systems are of considerable practical interest in problems of the functioning of instruments, aerospace tether systems, lifting-transport mechanisms, attractions, etc. (see [1, 2] and the bibliography given there). In practice, various methods of control can be implemented: external – by means of a force and/or the moment of forces about a fixed axis [1], inertial – by controlled displacement of the suspension point [1, 2], parametric – using a regulated change in the length of the suspension [2] or relative displacement of internal masses [2–4], etc.

In this paper we investigate the problem of time-optimal control of the plane oscillations and rotations of a pendulum-type system – a rigid body on a plane-parallel bifilar suspension (see Fig. 1). The control functions are relative displacements of the internal mass, regulated in acceleration or velocity (the "rotating swing" or "Czech swing" model). Note that the motions of a body on a bifilar suspension have specific properties.

1. FORMULATION OF THE PROBLEM

To fix our ideas and for simplicity we will consider a symmetrical form of the suspension using absolutely rigid rods of the same length l (see Fig. 1). The rigid body M can have an arbitrary mass distribution. The fixed hinges of the suspension are connected to the horizontal axis X and of an inertial system XY; the distance between them is d. The distance between the mobile hinges on the body M is also equal to d, so that the motions of the body are transnational. The rods which successively connect the hinges form a parallelogram. The centre of mass M of the body and all its points move along circles of radius 1 with fixed centres in the XY system. These motions and orientations of the axes of the rods relative to the vertical Y are defined by the angle φ.

We will further assume that the body M is the carrier and the carrier object of mass m is connected to it by holonomic non-stationary constraints. This object (in particular, a point mass) mass perform relative transnational motions. To describe the displacements we will introduce a system of coordinates xy, connected with the body, the x axis of which passes through the points of the mobile hinges, while

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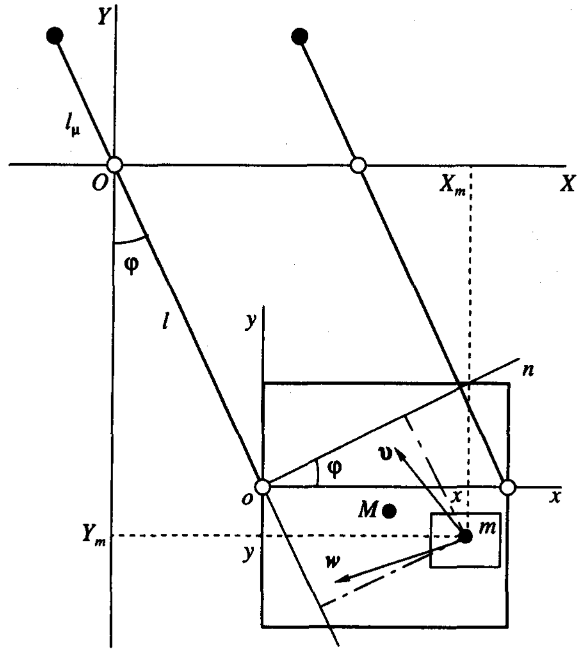

Fig. 1

the left hinge is the origin. In this system the point M (the centre of mass of the carrying body) is fixed and has constant coordinates x_m, y_m, while the centre of mass m of the mobile object is described by the coordinates x_m, y_m, i.e. by the vector $\mathbf{r}_m$, which can vary with time t under the control [2, 4].

The coordinates X_M, Y_M and X_m, Y_m of the points M and m in the inertial system XY can be represented by the expressions

$$\begin{aligned} X_M &= l\sin\varphi + x_M, \quad Y_M = -l\cos\varphi + y_M, \quad l, x_M, y_M = \text{const} \\ X_m &= l\sin\varphi + x, \quad Y_m = -l\cos\varphi + y, \quad x_m \equiv x, \quad y_m \equiv y \end{aligned} \tag{1.1}$$

which are independent of the parameter d. To simplify the notation the subscript m, indicating the mobile point, will henceforth be omitted. By differentiating expressions (1.1) we obtain the components of the velocities, on the basis of which we can calculate the kinetic energy K of the system, taking into account the total kinetic energy K_μ of the rotational motions of the suspension rods

$$\begin{aligned} K &= K_M + K_m + K_\mu, \quad K_M = \frac{1}{2}Ml^2\dot\varphi^2, \quad K_\mu = \frac{1}{2}I\dot\varphi^2 \\ K_m &= \frac{1}{2}m(l^2\dot\varphi^2 + 2l\dot\varphi v_n + v^2), \quad v_n = \dot x\cos\varphi + \dot y\sin\varphi, \quad v^2 = \dot x^2 + \dot y^2 \end{aligned} \tag{1.2}$$

here I is the total moment of inertia of the rods (with the counterweights, see below) about the fixed hinges, and v_n is the component of the relative velocity vector $\mathbf{v}$ of the point m, normal to the rods. To construct the equation of motion we must obtain the potential energy W and the Lagrange function L of the system

$$\begin{aligned} W &= W_M + W_m + W_\mu, \quad W_M = MgY_M, \quad W_m = mgY_m, \quad W_\mu = -\mu g l_\mu \cos\varphi \\ L &= K - W = \frac{1}{2}I^*\dot\varphi^2 + ml\dot\varphi v_n + \frac{1}{2}mv^2 - W, \quad I^* = (M+m)l^2 + I \end{aligned} \tag{1.3}$$

In expressions (1.3) we have used representations (1.1) for Y_M and Y_m and (1.2) for K.

Note that the kinetic energy K is a homogeneous quadratic form of the variables $\dot\varphi, \dot x, \dot y$. The potential energy W_μ (1.3) of the elements of the suspension is determined by the total mass μ and the reduced arm of the gravitational forces l_μ, which can take both positive and negative (or zero) values due to the presence of the counterweights. For simplicity we will assume that the distributions of the masses of the rods and the counterweights are such that the centres of mass lie on the axes connecting the points of the movable and fixed hinges. The potential energy W_m (1.3) of the point m is determined by the generalized coordinate φ and the variable quantity y, as given by (1.1).

We will further assume that the relative motion $\mathbf{r}(t)$ of the point m, i.e. the functions $x(t)$ and $y(t)$ are given. Neglecting possible perturbing factors, using the Lagrange function L (1.3) we obtain the equation of motion

$$\ddot{\varphi} + \nu^2 \sin\varphi = -\gamma w_n, \quad w_n = \ddot{x}\cos\varphi + \ddot{y}\sin\varphi, \quad w_l = -\ddot{x}\sin\varphi + \ddot{y}\cos\varphi$$
$$\gamma = ml/I^*, \quad \nu^2 = g(Ml + ml + \mu l_\mu)/I^*, \quad \nu^2 > 0 \tag{1.4}$$

Here w_n and w_l are the components of the vector of relative acceleration $\mathbf{w}$ of the point m, normal and parallel to the axes of the rods, respectively, and ν is the frequency of small oscillations of the bifilar pendulum when $w_n \equiv 0$. Equation (1.4) is identical in form with that obtained for a physical pendulum, the suspension point of which moves [1, 2]. It is natural to consider the functions $\ddot{x}(t)$ and $\ddot{y}(t)$ as control functions. Then, for the system described by Eq. (1.4) we can formulate and investigate interesting problems of the optimal control of the oscillatory and rotational motions. Using standard methods of the theory of optimal control, the variables φ, $\dot{\varphi}$, x, $\dot{x}$, y, $\dot{y}$ can be subjected to the required change by changing control functions $\ddot{x}$, $\ddot{y}$, i.e. the vector $\mathbf{w}$ of the relative acceleration of the point m.

The situation often arises in practical problems when the relative velocity $\mathbf{v}$ of displacement of the internal mass m (or the velocity of the suspension point [1, 2]) may change practically instantaneously in a certain limited region. This leads to impulsive controls and requires the development of special methods of solving the corresponding non-linear control and optimization problem. Impulsive controls lead to a discontinuous (piecewise-smooth) function $\varphi(t)$, but the function $\dot{\varphi}(t)$ will be continuous and piecewise-smooth with corner points (absolutely continuous). For the system considered, this difficulty can be overcome by introducing the Hamiltonian variables (φ, β)

$$\beta = \frac{\partial L}{\partial \dot{\varphi}} = I^*\dot{\varphi} + ml\upsilon_n$$
$$H = \beta\dot{\varphi} - L = \frac{\beta^2}{2I^*} - \frac{ml}{I^*}\beta\upsilon_n + \frac{(ml\upsilon_n)^2}{2I^*} - \frac{m\upsilon^2}{2} + W(\varphi, y) \tag{1.5}$$

Here β is the generalized momentum (the angular momentum), and H is the Hamilton function of the system. The equations of motion do not contain generalized (impulsive) functions and have the form

$$\dot{\varphi} = H'_\beta = \frac{\beta}{I^*} - \frac{ml}{I^*}\upsilon_n, \quad \upsilon_n = \dot{x}\cos\varphi + \dot{y}\sin\varphi, \quad \upsilon_l = -\dot{x}\sin\varphi + \dot{y}\cos\varphi$$
$$\dot{\beta} = -H'_\varphi = -\nu^2 I^* \sin\varphi + (ml/I^*)\beta\upsilon_l - (m^2l^2/I^*)\upsilon_n\upsilon_l \tag{1.6}$$

For system (1.6) it is natural to take the function $\dot{x}(t)$ and $\dot{y}(t)$ as the control functions, these can be piecewise-continuous, in particular bang-bang. The quantity υ_l has the meaning of the tangential component of the vector of the relative velocity, i.e. the projection of $\mathbf{v}$ onto the axis of the rod.

Note the main properties of the motion of a bifilar pendulum, described by the equations in Lagrange form (1.4) and Hamilton form (1.6). When $w_n \equiv 0$, i.e. $\mathbf{w} \equiv 0$, Eq. (1.4) has a first integral of the standard form

$$E = \frac{1}{2}I^*\dot{\varphi}^2 - g(Ml + ml + \mu l_\mu)\cos\varphi = \text{const} \tag{1.7}$$

which characterizes the total energy of the oscillations or rotations of the pendulum without taking into account the quantity mgy in W_m (1.3), where $y = y^0 + \dot{y}^0 t$. Using integral (1.7), the equation can be completely integrated in elliptic functions [1, 2]. If we put $\dot{y} = 0$, the function $E + mgy$, $y = \text{const}$, will also be a first integral.

When $\mathbf{v} \equiv 0$ system (1.6) has a first integral of the form (see (1.5))

$$H = \frac{1}{2}\beta^2/I^* + W(\varphi, y) = \text{const} \tag{1.8}$$

since $y = y^0 = \text{const}$. Using relation (1.8), this system can be completely integrated in terms of elliptic functions.

The quantity H represents the total energy of the oscillations or rotations of the pendulum, taking the term mgy into account.

The solution and investigation of problems of control of motions of a non-linear oscillatory system will entail well-known analytic and computational difficulties [1, 2]. These increase considerably when complex constraints are imposed on the control **w** or **v**, taking into account the phase constraints, for example, on the admissible positions $\mathbf{r}(t)$ of the point m, i.e. the coordinates $x(t)$, $y(t)$, and also when additional requirements are imposed on the final values of **r** and **v**. For applications, however, it is of considerable interest to construct simplified locally optimal [1] or quasi-optimal [1, 2] modes of control, which have a clearly expressed resonance form. The corresponding excitations must, in a certain sense, be weak, so that in a time interval equal to the period of the oscillations of rotations, a relatively small change occurs in the main parameters of the motion, for example, the energy, the oscillation amplitude, the rate of rotation, etc. In long intervals, containing many (in practice, several) periods, a considerable change in these characteristics of the motion of the pendulum (and of the mobile object) must occur that is quasi-optimal in the sense of a specified performance criterion.

This approach involves the use of asymptotic methods of optimal control [2], based on the maximum principle [5] and methods of separating motions (averaging) [6, 7]. Below we propose to use this approach to solve time-optimal type problems when controlling the rotational-oscillatory motions of systems (1.4) and (1.6). Typical regions (limitations) of the admissible values of **w** and **v** are considered: a rectangle (in particular, a square or a segment), an ellipse (in particular, a circle) or line segments, inclined at an arbitrary fixed angle to one of the coordinate axes. A numerical parameter is further introduced and the controlled systems are reduced to standard form [1, 2].

2. THE REDUCTION OF THE CONTROL PROBLEMS TO STANDARD FORM

To give the equations of motion of a dimensionless form with a small parameter we will introduce the argument θ and the unit of length ρ as follows: $\theta = \nu t, x = \rho\xi, y = \rho\eta$, where ξ, η are dimensionless relative coordinates of the point m, while the quantity ρ is chosen from additional conditions. The main requirement is that the control functions must be relatively small [2].

Thus, in the case of Eq. (1.4), as a result of these changes of variables, we obtain

$$\begin{aligned} &\ddot{\varphi} + \sin\varphi = -\varepsilon w_n, \quad \varepsilon = \gamma\rho, \quad 0 < \varepsilon \ll 1 \\ &w_n = \ddot{\xi}\cos\varphi + \ddot{\eta}\sin\varphi \quad (|w_n|, |\ddot{\xi}|, |\ddot{\eta}| \sim 1) \end{aligned} \tag{2.1}$$

The dots once again denote derivatives with respect to the argument $\theta = \nu t$. The function w_n (2.1) is obtained from w_n (1.4) by dividing by $\rho\nu^2$ (the old notation is retained here). The smallness of the numerical parameter ε is ensured by the ratio $ml\rho/I^* \ll 1$. A representation of Eq. (1.4) that is similar to (2.1) is obtained by introducing, instead of ρ, the unit of acceleration b. Then the parameter $\varepsilon = \gamma b/\nu^2$ and, in particular, $\varepsilon = \gamma l$ when $b = l\nu^2$, while the function w_n in (2.1) is obtained by dividing the initial one by b.

For system (1.6), by making the parameters dimensionless, we obtain the expressions

$$\begin{aligned} &\dot{\varphi} = \chi - \varepsilon\upsilon_n, \quad \chi = \beta/(I^*\nu), \quad \varepsilon = \gamma\rho = ml\rho/I^* \ll 1 \\ &\dot{\chi} = -\sin\varphi + \varepsilon\chi\upsilon_l - \varepsilon^2\upsilon_n\upsilon_l, \quad \upsilon_n = \dot{\xi}\cos\varphi + \dot{\eta}\sin\varphi, \quad \upsilon_l = -\dot{\xi}\sin\varphi + \dot{\eta}\cos\varphi \end{aligned} \tag{2.2}$$

Here χ is the normalized momentum, and the dimensionless components $|\upsilon_n|$, $|\upsilon_l| \sim 1$ (the old notation is retained for these). The quantities υ_n and υ_l are obtained by dividing the initial quantities by $\rho\nu$. If, instead of the unit of length ρ we introduce the unit of velocity υ, the small parameter $\varepsilon = \gamma\upsilon/\nu$ and, in particular, $\varepsilon = \gamma l$ when $\upsilon = l\nu$.

Equations (2.1) and (2.2) are reduced to the standard form of weakly controlled systems with rotating phase [1, 2]. We take as the slow variable the total energy E of the oscillations or rotations of the system (ignoring the quantity η, see above), due to the variables φ, $\dot{\varphi}$ for (2.1) and φ, χ for (2.2), respectively,

$$E = \frac{1}{2}\dot{\varphi}^2 + 1 - \cos\varphi, \quad E = \frac{1}{2}\chi^2 + 1 - \cos\varphi \tag{2.3}$$

The relation between φ and E, ψ (ψ is the phase of the unperturbed oscillations or rotations) is given by elliptic integrals [1, 2]. However, henceforth this will not be required, since when solving control problems in the first approximation in the parameter ε, averaging of the equations over ψ is carried out using integrals (2.3). The equation of the first approximation for E (after dropping quantities $O(\varepsilon^2)$ in the case of system (2.2)) is

$$\dot{E} = \varepsilon(\mathbf{f}, \mathbf{u}), \quad \mathbf{f} = \mathbf{f}_w(\varphi, \dot{\varphi}), \quad \mathbf{u} = \mathbf{w}; \quad \mathbf{f} = \mathbf{f}_\upsilon(\varphi, \chi), \quad \mathbf{u} = \mathbf{v} \tag{2.4}$$

The structure of the vector-function $\mathbf{f}$ (2.4) is determined by the type of control with respect to the acceleration (2.1) or the velocity (2.2). To simplify the operation of maximizing the corresponding Hamilton function of the problem of time-optimal control it is assumed that the regions U of admissible values of $\mathbf{u} \in U$ have the form of a rectangle $U^{(1)}$, an ellipse $U^{(2)}$, or an inclined segment $U^{(3)}$.

$$\begin{aligned} U^{(1)} &= \{\mathbf{u} = (u_1, u_2)^T: |u_1| \le a_1, |u_2| \le a_2\}, \quad a_{1,2} = \text{const} \\ U^{(2)} &= \{\mathbf{u} = (u_1, u_2)^T: (u_1/a_1)^2 + (u_2/a_2)^2 \le 1\}, \quad a_{1,2} = \text{const} \\ U^{(3)} &= \{\mathbf{u} = (u_1, u_2)^T: u_1 = u\cos\delta, u_2 = u\sin\delta, |u| \le a\}, \quad a, \delta = \text{const} \end{aligned} \tag{2.5}$$

The components u_1 and u_2 can be specified in various systems of coordinates. We will consider two methods: in a system xy connected with the body and in a system nl, which rotates, together with the rods, by an angle φ (see the figure) namely

$$\begin{aligned} &\mathbf{u} = \mathbf{w}_0 = (\ddot{\xi}, \ddot{\eta})^T \equiv (w_x, w_y)^T, \quad \mathbf{u} = \mathbf{v}_0 = (\dot{\xi}, \dot{\eta})^T \equiv (\upsilon_x, \upsilon_y)^T, \quad \mathbf{u} \in U_0^{(i)} \\ &\mathbf{u} = \mathbf{w}_\varphi = (w_n, w_l)^T, \quad \mathbf{u} = \mathbf{v}_\varphi \equiv (\upsilon_n, \upsilon_l)^T, \quad \mathbf{u} \in U_\varphi^{(i)}, \quad i = 1, 2, 3 \end{aligned} \tag{2.6}$$

Expression (2.6) have a clear geometrical content.

Hence, by representations (2.5) and (2.6) there is a correspondence $u_1 = w_x$, $u_2 = w_y$ or $u_1 = w_n$, $u_2 = w_1$ for the region U_0 or $U\varphi$ respectively; similarly $u_1 = \upsilon_x$, $u_2 = \upsilon_y$ or $u_1 = \upsilon_n$, $u_2 = \upsilon_1$.

The components w_n and w_l of the vector $\mathbf{w}_\varphi$ are defined in (1.4), while υ_n and υ_l of the vector $\mathbf{v}_\varphi$ are defined in (2.2). Using expressions (2.1) and (2.2) for the components of the vector $\mathbf{f}_w$ and $\mathbf{f}_\upsilon$, we obtain the expressions

$$\begin{aligned} &\mathbf{f}_{w_0} = (-\dot{\varphi}\cos\varphi, -\dot{\varphi}\sin\varphi)^T, \quad \mathbf{f}_{\upsilon_\varphi} = (-\sin\varphi, \chi^2)^T \\ &\mathbf{f}_{\upsilon_0} = (-(\chi^2 + \cos\varphi)\sin\varphi, \chi^2\cos\varphi - \sin^2\varphi)^T, \quad \mathbf{f}_{w_\varphi} = (-\dot{\varphi}, 0)^T \end{aligned} \tag{2.7}$$

The relation between the variables $\dot{\varphi}$, χ and E, φ is specified by the change of variables (2.3). The dependence on the phase ψ is implicit (in terms of elliptic functions and integrals). The equation for ψ, as was pointed out above, is not required [1, 2].

We will consider the problem of the time-optimal change in the energy E of the oscillations or rotations

$$E(0) = E^0, \quad E(\theta^f) = E^f, \quad \theta^f \to \min_{\mathbf{u}}, \quad \mathbf{u} \in U \tag{2.8}$$

according to Eq. (2.4). We will consider four types of control (2.6) and (2.7), for each of which there are three regions of admissible values of (2.5) (altogether 12 versions of the control problem). Note that the number of control modes is doubled (and equal to 24) due to the different description of the motion in the oscillation and rotation states. Hence, a detailed solution for all cases, which may only differ slightly, is hardly reasonable. In its theoretical and applied aspects, it is of interest to investigate the main properties of the controlled motions, and also to set up different control procedures.

In addition to the energy (or amplitude), with the appropriate conditions, we can take as the slow control variables the relative velocity $\mathbf{v}$ (for system (2.1)) or the relative position $\mathbf{r}$ (for (2.2)). These problems are extremely difficult to solve effectively. Hence, at the initial stage we will confine ourselves to investigating the simpler control problems (2.4)–(2.8). We can then determine the values of these additional variables and correct them without any appreciable change (in limits of the error of $O(\varepsilon)$) in the main variable [2].

It is of considerable interest to represent the mechanism for controlling the oscillations and rotations of a bifilar pendulum due to displacement of the mass m along a fixed fairly smooth curve $x = x(s)$, $y = y(s)$, including a closed curve. It is natural to take the acceleration $\ddot{s}$ or the rotate of change $\dot{s}$ of the parameter s of this curve as the control function for system (2.1) or (2.2) respectively. These problems require a separate discussion.

3. CONTROL OF THE MOTIONS BY MEANS OF REGULATED ACCELERATION

Consider problem (2.4)–(2.8) with $\mathbf{u} = \mathbf{w}$, i.e. $\mathbf{u} = \mathbf{w}_0 \in U_0^{(i)}$ or $\mathbf{u} = \mathbf{w}_\varphi \in U_\varphi^{(i)}$, according to the corresponding expressions (2.6) and (2.7). Its approximate solution with a relative error of $O(\varepsilon)$ is constructed using asymptotic methods of optimal control [1, 2]. It can be shown that the quasi-optimal control is locally optimal:

$$\mathbf{w}^* = -\sigma \underset{\mathbf{w}}{\operatorname{argmax}}(\mathbf{f}_w, \mathbf{w}), \quad \mathbf{w} \in U, \quad \sigma = \operatorname{sign}(E - E^f) \tag{3.1}$$

The components of the vector function $\mathbf{f}_w$ are defined in (2.7). The required control is obtained in the form of a synthesis, i.e. by the feedback principle, and requires highly accurate continuous measurements of the variables φ, $\dot{\varphi}$. The expression for $\mathbf{w}^*$ (3.1) can be constructed in explicit form for the regions $U_{0,\varphi}^{(i)}$ (2.5). The region $U_{0,\varphi}^{(1)}$, which has the form of a rectangle, leads to the functions

$$\begin{aligned} &w_x^* = a_1\sigma\operatorname{sign}(\dot{\varphi}\cos\varphi), \quad w_y^* = a_2\sigma\operatorname{sign}(\dot{\varphi}\sin\varphi), \quad \mathbf{w}^* \in U_0^{(1)} \\ &w_n^* = a_1\sigma\operatorname{sign}\dot{\varphi}, \quad -a_2 \le w_l^* \le a_2, \quad \mathbf{w}^* \in U_\varphi^{(1)} \end{aligned} \tag{3.2}$$

According to relations (3.2) the control w_l^* can be arbitrary within acceptable limits, since the equation of motion (2.4), as follows from expressions (2.7), is independent of w_1. Moreover, in the case of the limitation $U_\varphi^{(1)}$ the control w_n^* is equivalent to a constrained torque about a fixed axis [1, 2].

As in the case of an elliptic region $U^{(2)}$ (2.5), from relations (3.1) we obtain the expressions

$$\begin{aligned} &w_x^* = n_x a_1\sigma\operatorname{sign}\dot{\varphi}, \quad w_y^* = n_y a_2\sigma\operatorname{sign}\dot{\varphi}, \quad \mathbf{w}^* \in U_0^{(2)} \\ &n_x = h_x/h, \quad n_y = h_y/h, \quad h_x = a_1\cos\varphi, \quad h_y = a_2\sin\varphi, \quad h = |\mathbf{h}| \\ &w_n^* = a_1\sigma\operatorname{sign}\dot{\varphi}, \quad w_l^* \equiv 0, \quad \mathbf{w}^* \in U_\varphi^{(2)} \end{aligned} \tag{3.3}$$

Here n_x and n_y are components of the unit vector. As above, the control $\mathbf{w}^* \in U_\varphi^{(2)}$ is equivalent to a constrained torque. In the special case of a circular region $U_0^{(2)}$ (where $a_1 = a_2 = a$) the vector $\mathbf{w}^*$ is directed along the normal to the axes of the rods, i.e. collinear with the vector $\mathbf{w}^* \in U_\varphi^{(2)}$, while the control is equivalent to an axial torque.

The situation when the control vector $\mathbf{u} \in U_{0,\varphi}^{(3)}$ (2.5), i.e. the acceleration of the object m makes a constant angle $\pi/2 - \delta$ with the y and l axes, is extremely interesting from the theoretical point of view and useful in practice. It leads to the following expressions for the feedback control

$$\begin{aligned} &w_x^* = a\sigma\operatorname{sign}(\dot{\varphi}\cos(\varphi-\delta))\cos\delta, \quad w_y^* = a\sigma\operatorname{sign}(\dot{\varphi}\cos(\varphi-\delta))\sin\delta \\ &w_n^* = a\sigma\operatorname{sign}\dot{\varphi}|\cos(\varphi-\delta)|, \quad |\delta| \le \pi/2, \quad \mathbf{w}^* \in U_0^{(3)} \\ &w_n^* = a\sigma\operatorname{sign}\dot{\varphi}\cos\delta, \quad w_l^* = a\sigma\sin\dot{\varphi}\sin\delta, \quad \mathbf{w}^* \in U_\varphi^{(3)} \end{aligned} \tag{3.4}$$

The effectiveness of the control $\mathbf{u} \in U^{(3)}$ depends very much on the angle δ, particularly for small $|\varphi|$ (see expressions (3.4) and below). Moreover, when $\mathbf{w}^* \in U_\varphi^{(3)}$, the control is equivalent to a simple version of the control by means of a constrained torque, as is the case also for other constraints (see (3.2) and (3.3)).

An approximate investigation of the time-optimal change in the reduced energy E of the oscillatory or rotational motions of a pendulum can be carried out using Eq. (2.4), averaged over the phase φ, after substituting the expressions $\mathbf{f} = \mathbf{f}_{w_{0,\varphi}}(\varphi, \dot{\varphi})$ (2.7) and $\mathbf{w} = \mathbf{w}^*(\varphi, \dot{\varphi})$ (3.2)–(3.4). Here the variables φ and $\dot{\varphi}$ are assumed to be expressed in terms of E and ψ in accordance with the change of variables corresponding to the unperturbed motion ($\varepsilon = 0$), which can be represented using elliptic functions and integrals [1, 2]. This approach is extremely difficult to use; it is justified in the case of small (quasi-linear) oscillations. In the general case of non-linear oscillations and rotations, it is more effective to use the procedure of averaging the right-hand side "along the unperturbed trajectory" [1, 2, 7], i.e. using the relation $d\theta = d\varphi/\dot{\varphi}$ and subsequent integration over φ. The averaging scheme has the form

$$\langle w_n^* \dot{\varphi}\rangle_\psi \equiv \frac{1}{2\pi}\int_0^{2\pi} w_n^*\dot{\varphi}d\psi = \frac{1}{T}\int_0^T w_n^*\dot{\varphi}d\theta = \frac{1}{T}\oint w_n^* d\varphi \equiv W(E) \tag{3.5}$$

Depending on the mode of motion (oscillations for $E < 2$ or rotations for $E > 2$) the expressions for the period $T(E)$ and the integral over the closed contour (3.5), corresponding to the phase trajectory $\dot{\varphi} = \dot{\varphi}_0\ (E, \varphi)$, have different representations. For example, in the oscillation mode, the following formulae are obtained.

$$\begin{aligned}
&\oint w_n^* d\varphi = \int_{-\varphi_0}^{\varphi_0} [w_n^*(\varphi, \dot{\varphi}_0^+) - w_n^*(\varphi, \dot{\varphi}_0^-)]d\varphi, \quad \dot{\varphi}_0^\pm \equiv \sqrt{2}(E - 1 + \cos\varphi)^{1/2}\\
&T = T_\upsilon(E) = 4\mathbf{K}(k_\upsilon), \quad k_\upsilon \equiv \sqrt{E/2}, \quad 0 < k_\upsilon < 1\\
&E < 2, \quad \varphi_0 = \varphi_0(E) = \arccos(1 - E)
\end{aligned} \tag{3.6}$$

Here $\mathbf{K}(k)$ is the complete elliptic integral of the first kind with modulus $k = k_\upsilon$; the value for the amplitude φ_0 is taken in the first two quadrants: $0 < \varphi_0 < \pi$ $(0 < E < 2)$. The quadratures (3.6) are found in terms of elementary functions or elliptic integrals. Quasi-linear oscillations ($k_\upsilon \ll 1$) lead to much simpler expressions [1, 2].

In the rotation mode, we have the following formulae

$$\begin{aligned}
&\oint w_n^* d\varphi = \int_0^{\pm 2\pi} w_n^*(\varphi, \dot{\varphi}_0^\pm)d\varphi, \quad T = T_r(E) = 2k_r\mathbf{K}(k_r)\\
&k_r = \sqrt{2/E}, \quad 0 < k_r < 1, \quad E > 2, \quad \dot{\varphi}_0^\pm \gtrless 0, \quad |\varphi| < \infty
\end{aligned} \tag{3.7}$$

The signs ± correspond to positive or negative (counter-clockwise or clockwise) rotational motion of the body M (rotations of the rods). To fix our ideas we will consider the case of positive rotations (the plus sign). The quadratures (3.7) can be calculated in terms of elementary functions or elliptic integrals. In the case of fast rotations $k_r \ll 1$ ($E \gg 1$), expressions (3.7) are simplified considerably.

As as result, according to relations (3.5)–(3.7), the averaged right-hand side of the equation for E (2.4) is expressed in terms of elliptic integrals and elementary functions, containing the unknown E. This enables a "slow time" to be introduced – the argument $\tau = \varepsilon\theta$, which enables the variables E and τ to be separated and enables approximate values of the functional (the optimal time of motion) and the energy to be determined with a relative error of $O(\varepsilon)$

$$\tau = \int_{E_0}^{E} \frac{d\zeta}{W(\zeta)}, \quad \tau^f = \varepsilon\theta^f = \int_{E_0}^{E^f} \frac{dE}{W(E)}, \quad W(E) = \langle w_n^* \dot{\varphi}_0\rangle_\psi \tag{3.8}$$

Note that the sign of the function $W(E)$ is the same as the sign of the different $E - E^0 \neq 0$. The relatively simple case mentioned above, which leads to a constrained equivalent torque (see (3.2)–(3.4)), is described in the oscillation and rotation modes by the functions W_υ and W_r respectively

$$\begin{aligned}
&W_\upsilon(E) = -\gamma_\upsilon \operatorname{sign}(E - E^f)(\mathbf{K}(\sqrt{E/2}))^{-1}\arccos(1 - E), \quad 0 < E^0, \quad E < 2\\
&W_r(E) = -\gamma_r \operatorname{sign}(E - E^f)(\sqrt{2/E}\mathbf{K}(\sqrt{2/E}))^{-1}, \quad 2 < E^0, \quad E < \infty
\end{aligned} \tag{3.9}$$

Here γ_υ and γ_r are numerical coefficients, determined by the values of the parameters a_1 and a (a_2 has no effect on the solution). Since $\gamma_{\upsilon,r}$ are proportional to a_1 or a, it follows from relations (3.9) that the system is uncontrollable when $a_1 = 0$ or $a = 0$ in the first approximation in ε considered. This fact is of interest from the view-point of mechanics.

For the general situation $\gamma_{\upsilon,r} > 0$; by introducing the argument $\tau' = \gamma_{\upsilon,r}\tau$ the system can be written in a form which does not contain the parameters. This enables us to construct a unified relation $E = E(\tau', E^0)$ [1, 2]. An analytical and numerical investigation of the controlled oscillations and rotations presents no difficulties. In particular, when $E \ll 1$ (quasi-linear oscillations) and $E \gg 1$ (fast rotations) the functions $W_{\upsilon,r} \sim \sqrt{E}$, which confirms that E depends quadratically on τ'. When $E \sim 1$, particularly in the neighbourhood of $E = 2$, i.e. the separatrices in the phase plane $(\varphi, \dot\varphi)$, numerical calculations can be carried out. They also show that, on the whole, the relation $E(\tau, E^0)$ is "close" to the segment of a parabola for all τ when $E^0 \neq 2$; the neighbourhood $E^0 = 2$ requires additional investigation.

If $E \to 2$, then $k_{\upsilon,r} \to 1$ from below, and the periods of the oscillations or rotations $T_{\upsilon,r} \to \infty$. This fact makes it difficult to use and justify the averaging method [8, 9]. Moreover, the right-hand side of the formally averaged equation (2.4) tends to zero as $E \to 2$: $W(2) = 0$, i.e. the rate of transition through the separatrice is zero. However, this singularity is integrable, which does not lead to "sticking" and, moreover, the error of the averaging method is a quantity $O(\varepsilon\ln\varepsilon^{-1})$ for $\theta \sim 1/\varepsilon$. Hence, the averaging method turns out to be applicable for arbitrary admissible values of the reduced energy E. When integrating numerically in a small neighbourhood of the value $E = 2$, for example, when $E^f = 2$, one can use the asymptotic form [1]

$$\gamma\tau \approx \frac{|E-2|}{2\pi}\left(1 + \ln\frac{32}{|E-2|}\right), \quad \gamma = \gamma_{\upsilon,r} \tag{3.10}$$

Formula (3.10) enables one to "transfer" from the oscillatory mode to the rotational mode and vice versa. The unified curve $E(\tau', 2)$ is constructed, and a complete solution of the optimal control problem is obtained in the first approximation in ε [1].

We will briefly consider the optimum evolution of the energy E of the oscillations or rotations, when the regions of control variables u are specified in a system of coordinates xy connected with the body M as given by relations (3.2)–(3.4): $\mathbf{w}^* \in U_0^{(i)}$, $(i = 1, 2, 3)$. The expressions for the averaged right-hand sides in the oscillation mode $W_\upsilon(E)$ and the rotation mode $W_r(E)$, respectively, when $i = 1$ ($U_0^{(1)}$ is a rectangle) have the form

$$\begin{aligned} W_\upsilon(E) &= -\mathrm{sign}(E-E^f)(\mathbf{K}(\sqrt{E/2}))^{-1}[a_1(2E-E^2)^{1/2} + a_2E], \quad 0 < E \le 1 \\ W_\upsilon(E) &= -\mathrm{sign}(E-E^f)(\mathbf{K}(\sqrt{E/2}))^{-1}[a_1(2-(2E-E^2)^{1/2}) + a_2E], \quad 1 \le E < 2 \\ W_r(E) &= -2\,\mathrm{sign}(E-E^f)(a_1+a_2)(\sqrt{2/E}\mathbf{K}(\sqrt{2/E}))^{-1}, \quad E > 2 \end{aligned} \tag{3.11}$$

The function $W_\upsilon(E)$ is continuous and smooth everywhere, including the point $E = 1$ where $(W'_\upsilon(1-0) = W'_\upsilon(1+0))$. The averaged system is integrable according to relations (3.8) and (3.11) and can be investigated by analytical and numerical methods. Its behaviour is qualitatively similar to that of the system investigated above when $a_1 > 0$. If $a_1 = 0$, controllability occurs everywhere except the rest position $E = 0$. This means that when $E^0 > 0$, $E^f > 0$ a finite time τ^f (3.8) is required; when $E^0 = 0$ or $E^f = 0$ the control problem is unsolvable in the interval $\theta^f \sim \varepsilon^{-1}$. As described above, the asymptotic forms of the solution when $E \ll 1$ $(a_1 > 0)$, $E \approx 2$ (see (3.10)) and $E \gg 1$ are investigated. Instead of the argument τ one can introduce $\tau' = a_0\tau$, $a_0 = \sqrt{a_1^2 + a_2^2}$ and construct a unified family of curves $E(\tau', E^*, \alpha)$, where E^* is a fixed value (in particular $E^* = 2$); the parameter α is defined by the relation $\cos\alpha = a_1/a_0$, $\sin\alpha = a_2/a_0$. Note that, in the rotation mode $E > 2$, the effectiveness of both components w_x^* and w_y^* is the same and is determined by the value of the limitation a_1 and a_2 respectively.

We will also briefly consider the case of an elliptic region $U_0^{(2)}$, see (2.5) and (3.3). Averaging, according to relations (3.5), leads to expressions for the right-hand sides of $W_{\upsilon,r}(E)$ of the form

$$\begin{aligned} W_\upsilon(E) &= -a_1\,\mathrm{sign}(E-E^f)(\mathbf{K}(\sqrt{E/2}))^{-1}F(\varphi_0(E), k_1), \quad a_1 \ge a_2 \\ W_\upsilon(E) &= -a_2\,\mathrm{sign}(E-E^f)\left(\mathbf{K}\left(\frac{\sqrt{E}}{2}\right)\right)^{-1}\left[E_*(k_2) - F\left(\frac{\pi}{2} - \varphi_0(E), k_2\right)\right], \quad a_2 \ge a_1 \\ &k_1^2 = (a_1^2 - a_2^2)a_1^{-2}, \quad k_2^2 = (a_2^2 - a_1^2)a_2^{-2}, \quad 0 \le k_{1,2} \le 1 \\ W_r(E) &= -2a_{1,2}\,\mathrm{sign}(E-E^f)(\sqrt{2/E}\mathbf{K}(\sqrt{2/E}))^{-1}E_*(k_{1,2}) \end{aligned} \tag{3.12}$$

Here $\mathbf{K}$ and F are the complete and incomplete elliptic integrals of the first kind, respectively, with the corresponding moduli $k_{\upsilon,r}$ and $k_{1,2}$, and E_* is the complete elliptic integral of the second kind with modulus k_1 and k_2. In particular, when $a_1 = a_2 = a$ ($U_0^{(2)}$ is a circle) the moduli $k_{1,2} = 0$ and we have the case of a torque constrained in magnitude, see (3.9) and (3.10). The qualitative features of the controlled motion when $a_1 = 0$ (the ellipse degenerates into a segment $(-a_2, a_2)$ along the y axis) repeat those investigated above for a rectangular region $U_0^{(1)}$. For $E \ll 1$ and $E \gg 1$ the asymptotic forms derived above (the quadratic dependence on τ) hold; when $E \approx 2$ the asymptotic form (3.10) and the commentaries on it hold.

We will investigate very briefly the control problem in the case of the "region" $U_0^{(3)}$ – a line segment $2a$ inclined at the angle δ to the x axis. This control can be regarded as the result of the projection of the components $w_{x,y}$ of the vector $\mathbf{w}_* \in U_0^{(1)}$ onto this straight line, see (3.4). After averaging, in accordance with relations (3.6) and (3.7), expressions are obtained for $W_{\upsilon,r}(E)$ of the form (3.11) for $a_1 = a|\cos\delta|$, $a_2 = a|\sin\delta|$. Taking these representations for $a_{1,2}$ into account, we can carry out a complete analytical and numerical investigation and construct a unified family of solutions $E(\tau', E^*, \delta)$. This complete solution and analysis of the dependence on δ, E^0, E^f of all the parameters of motion were presented previously in [1, 2]. The qualitative relation between the solutions in the case of different regions $U_0^{(i)}$ ($i = 1, 2, 3$) (2.5) of admissible values of the control $\mathbf{w}$ is of considerable interest.

Note that the possible drift of the object m with respect to the body M can be eliminated either during the process of control of the oscillations or rotations or when it is completed by means of smooth (non-resonant) actions [1, 2]. An investigation of the more general class of problems of the simultaneous control of the motions of the body M and the object m may be of interest, but it presents considerable computational difficulties.

4. CONTROL OF THE MOTIONS BY MEANS OF A REGULATED VELOCITY

We will now consider the less-studied problem (2.4)–(2.8) when $\mathbf{u} = \mathbf{v}$, i.e. $\mathbf{v}_0 \in U_0^{(i)}$ or $\mathbf{v}_\varphi \in U_\varphi^{(i)}$, ($i = 1, 2, 3$). An approximate solution, with relative error $O(\varepsilon)$ can be constructed using asymptotic methods of optimal control [1, 2], similar to the constructions in Section 3. The quasi-optimal control is locally optimal and is defined by the relation

$$\mathbf{v}^* = -\sigma \operatorname*{argmax}_{\mathbf{v}}(\mathbf{f}_\upsilon, \mathbf{v}), \quad \mathbf{v} \in U = U_{0,\varphi}^{(i)}, \quad \sigma = \operatorname{sign}(E - E^f) \tag{4.1}$$

The vector function $\mathbf{f}_\upsilon$ is defined in (2.7). The required functions $\mathbf{v}_0$ and $\mathbf{v}_\varphi$ can be obtained in the feedback form and require highly accurate continuous measurements of the phase coordinates φ and $\dot\varphi$.

For the rectangular region $U_{0,\varphi}^{(1)}$ (2.5), from relation (4.1) we obtain expressions for $\mathbf{v}_{0,\varphi}$ and for the right-hand sides of Eq. (2.4) (the subscript υ is omitted for brevity)

$$\begin{aligned}
&\upsilon_{x,y} = -a_{1,2}\sigma \operatorname{sign} f_{x,y}, \quad (\mathbf{f}, \mathbf{v})^0 = -\sigma(a_1|f_x| + a_2|f_y|)\\
&f_x = -(\chi^2 + \cos\varphi)\sin\varphi, \quad f_y = \chi^2\cos\varphi - \sin^2\varphi\\
&\upsilon_{n,l} = -a_{1,2}\sigma \operatorname{sign} f_{n,l}, \quad (\mathbf{f}, \mathbf{v})^\varphi = -\sigma(a_1|\sin\varphi| + a_2\chi^2)\\
&f_n = -\sin\varphi, \quad f_l = \chi^2, \quad \sigma = \operatorname{sign}(E - E^f)
\end{aligned} \tag{4.2}$$

Expressions (4.2) are much more complex than (3.2), and a further numerical-analytical investigation, in particular, averaging similar to (3.5)–(3.7), is extremely difficult. The analysis can be simplified somewhat in the case of small oscillations ($|\varphi| \ll 1$) or fast rotations ($|\chi| \gg 1$).

For the elliptic region $U_{0,\varphi}^{(2)}$ (2.5), the determination of the controls and the right-hand sides of Eq. (2.4) leads to formulae of the same kind

$$\begin{aligned}
&\upsilon_{x,y} = -a_{1,2}\sigma n_{x,y}, \quad n_{x,y} = h_{x,y}/h^0, \quad h_{x,y} = a_{1,2}f_{x,y}\\
&h^0 = (h_x^2 + h_y^2)^{1/2}, \quad (\mathbf{f}, \mathbf{v})^0 = -\sigma h^0(\varphi, \chi)\\
&\upsilon_{n,l} = -a_{1,2}\sigma n_{n,l}, \quad n_{n,l} = h_{n,l}/h^\varphi, \quad h_{n,l} = a_{1,2}f_{n,l}\\
&h^\varphi = (h_n^2 + h_l^2)^{1/2}, \quad (\mathbf{f}, \mathbf{v})^\varphi = -\sigma h^\varphi(\varphi, \chi)
\end{aligned} \tag{4.3}$$

The function $\mathbf{f}^0$ and $\mathbf{f}^\varphi$ have the form (4.2). Expressions (4.2) are extremely complex and do not allow of explicit averaging in terms of tabulated functions. As above, they can be simplified considerably when analysing quasi-linear oscillations or rapid rotations.

For the one-dimensional region $U_{0,\varphi}^{(3)}$ (2.5) the synthesis of the approximately optimal controls (4.1) and the right-hand sides of Eq. (2.4) have the form

$$\begin{aligned} &v_{x,n} = -\sigma u^{0,\varphi}\cos\delta, \quad v_{y,l} = -\sigma u^{-0,\varphi}\sin\delta \\ &u^{0,\varphi} = a\,\mathrm{sign} f_*^{0,\varphi}, \quad f_*^{0,\varphi} = f_{x,n}\cos\delta + f_{y,l}\sin\delta, \quad (\mathbf{f},\mathbf{v})^{0,\varphi} = -a\sigma|\mathbf{f}_*^{0,\varphi}| \end{aligned} \tag{4.4}$$

In the general case, the analytic procedure of averaging is impossible in terms of known tabulated functions. When $E \sim 1$ the averaging can be carried out taking into account the relation $\chi^2 = 2(E - 1 + \cos\varphi)$ and the differential relation $d\theta = d\varphi/\chi$, similar to scheme (3.5)

$$\langle \mathbf{f},\mathbf{v}\rangle_\psi \equiv \frac{1}{2\pi}\int_0^{2\pi}(\mathbf{f},\mathbf{v})d\psi = \frac{1}{T}\int_0^T(\mathbf{f},\mathbf{v})d\theta = \frac{1}{T}\oint(\mathbf{f},\mathbf{v})\frac{d\varphi}{\chi} \equiv V(E) \tag{4.5}$$

An analysis of the function $V(E)$ (4.5) indicates that the qualitative features of the controlled motion when $E \sim 1$ are similar to those established above in Section 3 for "acceleration control". The system turns out to be controllable for all $E \geq 0$ if $a_1 > 0$, $|\delta| < \pi/2$; when $a_1 = 0$ or $|\delta| = \pi/2$ for $E^0 = 0$ or $E^f = 0$, controllability does not occur in the interval $\tau \sim 1$ ($\theta \sim \varepsilon^{-1}$). The transition of the phase trajectory φ, χ through the separatrice $E = 2$ occurs according to the asymptotic form of the type (3.10).

In the general situation one can construct unified single-parameter families by introducing the argument $\tau' = a_0\tau$ for (4.2) and (4.3) or $\tau' = a\tau$ for (4.4) with the angle parameter α or δ respectively (see Section 3). The case of small (quasi-linear) oscillations of the body M allows of considerable modification and generalization in formulating the problem, taking the relative position (x, y) of the object m into account. It can be investigated by asymptotic and numerical methods similar to the previously investigated problem of the control of the oscillations of a pendulum with velocity-controlled displacements of the suspension point [1, 2].

We will consider another limiting case of controlled motion, when $|\chi| \gg 1$ ("rapid rotations"), i.e. $E \sim \chi^2 \gg 1$. Expressions (4.2)–(4.4) for the control $\mathbf{v}$ can be simplified considerably using the perturbation method. One can take the quantity $\lambda = 1/E^f \ll 1$ as the small parameter λ, after normalising the variable E in Eq. (2.4). With a relative error of $O(\lambda)$ for the controls and the right-hand sides one obtains representations which have a clear mechanical meaning and can be simply used for the practical control of a rotating swing in the case of comparatively rapid rotations.

For a region of rectangular form $U^{(1)}$, instead of (4.2) we used expressions of the first approximation in λ for the controls and for the right-hand sides, and also for the average (4.5), of the form

$$\begin{aligned} &v_x = a_1\sigma\,\mathrm{sign}\sin\varphi, \quad v_y = -a_2\sigma\,\mathrm{sign}\cos\varphi, \quad (\mathbf{f},\mathbf{v})^0 = -2\sigma E(a_1|\sin\varphi| + a_2|\cos\varphi|) \\ &v_n \equiv 0, \quad v_l = -a_2\sigma, \quad (\mathbf{f},\mathbf{v})^\varphi = -2a_2\sigma E, \quad \mathbf{v}_{0,\varphi} \in U_{0,\varphi}^{(1)} \\ &V_0(E) = -(4/\pi)(a_1 + a_2)\sigma E, \quad V_\varphi(E) = -2a_2\sigma E \end{aligned} \tag{4.6}$$

It follows from relations (4.6) that for the region $U_0^{(1)}$ the controls v_x and v_y are equally effective; for the region $U_\varphi^{(1)}$ the control v_l is considerably more effective than v_n (in contrast to the case of small oscillations).

Constraints in the form of the elliptic region $U^{(2)}$ (see 3.12))

$$\begin{aligned} &v_x = a_1\sigma h_x/h, \quad v_y = -a_2\sigma h_y/h \\ &h_x = a_1\sin\varphi, \quad h_y = a_2\cos\varphi, \quad h = |\mathbf{h}|, \quad (\mathbf{f},\mathbf{v})^0 = -2\sigma E h \\ &v_n \equiv 0, \quad v_l = -a_2\sigma, \quad (\mathbf{f},\mathbf{v})^\varphi = -2a_2\sigma E, \quad \mathbf{v}_{0,\varphi} \in U_{0,\varphi}^{(2)} \\ &V_0(E) = -(4/\pi)a_{1,2}E_*(k_{1,2})\sigma E \quad (a_1 \lesseqgtr a_2), \quad V_\varphi(E) = -2a_2\sigma E \end{aligned} \tag{4.7}$$

lead to similar expressions instead of (4.3).

A comparison of expressions (4.6) and (4.7) for V_φ shows that they are identical in the first approximation in λ; the coefficient in V_0 (4.6) is greater than in V_0 (4.7), since the region $U^{(1)}$ is "wider" than $U^{(2)}$ for the same a_1 and a_2. Like (4.6) the controls υ_x and υ_y (4.7) possess the same effectiveness; the control υ_1 is much more effective (by an order of magnitude with respect to λ) than υ_n. For the one-dimensional region $U^{(3)}$ the required expressions

$$\begin{aligned} &\upsilon_x = a\sigma\,\mathrm{sign}(\cos\delta\sin\varphi), \quad \upsilon_y = -a\sigma\,\mathrm{sign}(\sin\delta\cos\varphi) \\ &(\mathbf{f},\mathbf{v})^0 = -2a\sigma E(|\cos\delta\sin\varphi| + |\sin\delta\cos\varphi|), \quad \mathbf{v}\in U^{(3)}_{0,\varphi} \\ &\upsilon_n \equiv 0, \quad \upsilon_l = -a\sigma\,\mathrm{sign}(\sin\delta), \quad (\mathbf{f},\mathbf{v})^\varphi = -2a\sigma E|\sin\delta| \\ &V_0(E) = -(4/\pi)a(|\cos\delta| + |\sin\delta|)\sigma E, \quad V_\varphi(E) = -2a|\sin\delta|\sigma E \end{aligned} \tag{4.8}$$

follow from relations (4.1) and (4.5).

The functions V_0 and V_φ (4.8) are analysed and compared with the other functions $V_{0,\varphi}$ (4.6) and (4.7) in the same way as above.

The averaged equations can be integrated in an elementary way

$$E = E^0\exp(-c\sigma^0\tau), \quad \tau^f = c^{-1}|\ln(E^0/E^f)|, \quad \sigma^0 = \mathrm{sign}(E^0 - E^f) \tag{4.9}$$

The coefficient c in expressions (4.9) takes different values, corresponding to expressions (4.6)–(4.8). A qualitative feature of the controlled motions of the system under rapid rotation conditions by means of a regulated change in the relative velocity $\mathbf{v}$ is the exponential change of the energy E with time. This fact has not been pointed out previously, although corresponding equations of the controlled motion of a pendulum with a regulated suspension length were obtained in [4], from which this qualitative conclusion follows. Note that, in the case of oscillations, it is possible to obtain a parabolic law of the change in the energy with time; on passing through the separatrice this law is close to a linear function of time. Moreover, it follows from relations (4.6)–(4.8) that the mean $\langle \mathbf{v}_0\rangle_\psi = 0$, i.e. the relative drift of the object m with respect to the body M is small for the control $\mathbf{v}_0 \in U_0^{(i)}$. The control $\mathbf{v}_\varphi \in U_\varphi^{(i)}$ leads to a considerable relative displacement, which may be unacceptable when solving applied problems.

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